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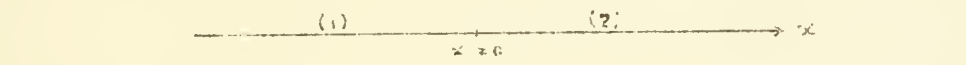
A SCHEME FOR THE COMPUTATION OF SHOCK WAVES IN GASES AND FLUIDS

By
Hermann Weyl

Temperatures in shock waves mount so high that the assumption of constant heat, or, what amounts to the same, of the linearity of the function f which expresses the energy U per mol in terms of temperature, is only a rough approximation. Instead we are going to treat f as an empirically given function. In order to obtain a clear-cut algebraic algorithm one is then forced to express everything in terms of the temperature. It turns out that ideal gases and fluids which obey Tammann's equation of state can be tackled by exactly the same scheme.

A. One-dimensional Shock

§1. Ideal gas. Let v, p, T denote velocity



pressure, absolute temperature, and V, U, $i = U + pV$, η volume, internal energy, heat content, entropy per mol. $\rho = V^{-1}$ is the density. We choose a coordinate x so that the shock front rests at $x = 0$. The problem is then stationary. The indices 1 and 2 refer to the states of the gas before and after the shock, $x < 0$ and $x > 0$ respectively. The equations relating the states 1 and 2 are as follows

$$\text{a)} \qquad \rho_1 v_1 = \rho_2 v_2 \ (= M) \ ,$$

$$\text{b)} \qquad M v_1 + p_1 = M v_2 + p_2 \ ,$$

$$\text{c)} \qquad i_1 + \tfrac{1}{2} v_1^2 = i_2 + \tfrac{1}{2} v_2^2 \ .$$

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The law of increasing entropy requires

$$d^*) \qquad M(\eta_2 - \eta_1) > 0 \quad .$$

We determine the direction of the positive x-axis by $M > 0$. Then the shock front has the negative velocity $-v_1$ with respect to the gas in the state (1), i.e. the front moves <u>into</u> (1), or (1) is indeed the <u>initial</u> state in which the gas finds itself before it is reached by the shock. Inequality d*) now reads

$$d) \qquad \Delta\eta = \eta_2 - \eta_1 > 0.$$

For an ideal gas we have the relation $p = RT\rho$, and U is a monotone increasing function $R.f(T)$ of T alone. Therefore

$$i = R.\phi(T) , \qquad \phi(T) = f(T) + T \quad .$$

In b) we replace p by $RT\rho$ and then use a) to substitue $\frac{M}{v}$ for ρ. Supposing the temperatures T_1 and T_2 to be given, we obtain the following two equations for v_1 and v_2.

$$(1) \qquad v_1 + \frac{RT_1}{v_1} = v_2 + \frac{RT_2}{v_2} \quad ,$$

$$(2) \qquad R(\phi(T_2) - \phi(t_1)) = \tfrac{1}{2}(v_1^2 - v_2^2) \quad .$$

It is clear that they detemine v_1, v_2 except for a common sign. Hence we shall obtain unique values for the ratio v_1/v_2 and the squares v_1^2, v_2^2. By writing $v\sqrt{R}$ instead of v we effect the normalization $R = 1$.

Multiply (1),

$$v_1 - v_2 = \frac{T_2}{v_2} - \frac{T_1}{v_1}$$

by $v_1 + v_2$ and use (2) :

$$2(\phi(T_2) - \phi(T_1)) = (T_2 - T_1) + (T_2\frac{v_1}{v_2} - T_1\frac{v_2}{v_1}).$$

Introducing the function

$$g(T) = 2\phi(T) - T = 2f(T) + T$$

we get for the quantities

$$D_1 = T_1\frac{v_2}{v_1}, \qquad D_2 = T_2\frac{v_1}{v_2}$$

the equations

$$(3) \qquad D_2 - D_1 = g(T_2) - g(T_1), \qquad D_1D_2 = T_1T_2 .$$

To fix the ideas suppose $T_1 < T_2$. (3) is nothing but the simple quadratic problem to transform a rectangle of the sides T_1, T_2 into one of equal area whose sides have the increased difference

$$g(T_2) - g(T_1) = (T_2 - T_1) + 2(f(T_2) - f(T_1)) > T_2 - T_1 .$$

Hence $D_1 < T_1 < T_2 < D_2$. (Were $T_2 < T_1$ we should have $D_2 < T_2 < T_1 < D_1$.) With the "displaced temperatures" D_1, D_2 we find

$$(4) \qquad k^{-1} = \frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{D_1}{T_1}, \quad k = \frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{D_2}{T_2} ;$$

$$\frac{p_1}{p_2} = \frac{D_1}{T_2}, \qquad \frac{p_2}{p_1} = \frac{D_2}{T_1} ;$$

and after some elementary trans formations the equation (2)

and $$\frac{T_2}{T_1}\left(\frac{v_1}{v_2}\right)^2 = \frac{D_2}{D_1}$$

yield

$$v_1^2 = \frac{D_2 - T_1}{T_1 - D_1}\cdot T_1 = \frac{1}{2}\,\frac{D_2^2 - T_1^2}{f(T_2) - f(T_1)} ,$$

$$v_2^2 = \frac{T_2 - D_1}{D_2 - T_2} \cdot T_2 = \frac{1}{2} \frac{T_2^2 - D_1^2}{f(T_2) - f(T_1)} \quad .$$

In this way all relevant quantities are expressed in terms of T_1 and T_2 .

The entropy η is given by

$$\eta/R = \int^T \frac{f'(T)}{T} dT - \log \rho \quad ,$$

thus

$$\frac{1}{R}(\eta_2 - \eta_1) = \int_{T_1}^{T_2} \frac{f'(T)}{T} dT - \log \frac{\rho_2}{\rho_1} \quad \text{or}$$

$$\frac{\Delta \eta}{R} = \eta(T_1, T_2)$$

where

$$\eta(T_1, T_2) = \int_{T_1}^{T_2} \frac{f'(T)}{T} dT - \log \frac{D_2}{T_2} = \int_{T_1}^{T_2} \frac{f'(T)}{T} dT + \log \frac{D_1}{T_1} \quad .$$

This function $\eta(T_1, T_2)$ is clearly skew-symmetric. We make the hypothesis which is fulfilled for most if not all gases in nature:

(H_1): $\eta(T_1, T_2)$ has the sign of $T_2 - T_1$.

For a small difference of T_1 , T_2 an easy calculation yields

$$\eta(T_1, T_2) = \frac{1}{6} Wf(T) \cdot \left(\frac{\Delta T}{T}\right)^3 \quad \text{with } T_1 = T - \frac{1}{2}\Delta T, \; T_2 = T + \frac{1}{2}\Delta T,$$

where

$$(5) \qquad Wf = \frac{1}{2}[f'(f' + 1)(2f' + 1) - f''T] \quad .$$

Hence our assumption of the monotony of f and (H_1) imply

$$Df = f' > 0, \qquad Wf > 0.$$

Question: Do these differential relations vice versa imply (H_1) ? We shall see that the expression Wf occurs again and again in our investigation.

Notice that $\Delta\eta$ is of the order $(\Delta T)^3$ for a small difference $\Delta T = T_2 - T_1$. On the basis of (H_1) the inequality d) is equivalent to $T_2 > T_1$: the shock raises temperature and density (condensation shock).

§2. Fluid. In Tammann's equation of state

$$(p + K)(V - b) = CT \qquad (b, K, C \text{ constants})$$

we introduce

$$p' = p + K, \qquad V' = V - b.$$

The fundamental thermodynamical equation for virtual variations

$$T\delta\eta = \delta U + p\delta V$$

gives

$$\delta\eta = \frac{\delta U}{T} + (p' - K)\frac{\delta V'}{T} = \frac{\delta(U - KV')}{T} + C\frac{\delta V'}{V'} .$$

Consequently $\frac{\delta(U - KV')}{T}$ is a total differential, or $U - KV'$ is a function $C.f(T)$ of T alone:

$$U = KV' + C.f(T),$$

$$i = (pV' + bp) + KV' + C.f(T)$$

$$= bp + p'V' + Cf(T) = bp + C.\phi(T) .$$

$Mv + p$ has the same value for the states 1 and 2, and so has $i + \frac{1}{2}v^2$. By subtracting b times the first expression from the second, one gets

$$C.\phi(T) + \tfrac{1}{2}v^2 - Mbv ,$$

or after replacing v by MV on the basis of a),

$$v_1/V_1 = v_2/V_2 = M,$$

one then finds the following two quantities

$$M^2V + p, \qquad C.\phi(T) + \tfrac{1}{2}M^2[(V - b)^2 - b^2]$$

cross the shock front unchanged. Hence the same is true for

$$M^2V' + p' = M^2V' + \frac{CT'}{V'} \quad \text{and}$$

$$C.\phi(T) + \tfrac{1}{2} M^2V'^2 .$$

Setting $MV' = v'$ we get the same equation (1) and (2) as before:

$$v_1' + \frac{CT_1}{v_1'} = v_2' + \frac{CT_2}{v_2'} ,$$

$$C.\phi(T_1) + \tfrac{1}{2}v_1'^2 = C.\phi(T_2) + \tfrac{1}{2} v_2'^2 .$$

After v_1', v_2' are determined from them, the relation

$$M = v_1'/V_1' = \frac{v_1'}{V_1' - b}$$

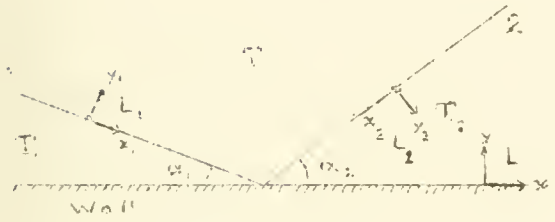
gives the value of M if the specific volume V_1 of the fluid in its initial state 1 is known besides the temperature T_1. Then V_2, v_1, v_2 follow from

$$\frac{V_2 - b}{V_1 - b} = \frac{v_1'}{v_2'} ; \quad v_1 = v_1' + Mb = v_1' \cdot \frac{V_1}{V_1 - b} ,$$

$$v_2 - v_1 = v_2' - v_1' .$$

B. Reflection of a Shock Wave

1. The formulas. In this two-dimensional problem we have two shock fronts 1 and 2 at rest, which meet on the reflecting wall. The components (x,y) and (x_1,y_1) of any vector in the coordinate systems L and L_1 (see figure) are connected by



$$x_1 = a_1x - b_1y, \qquad x = a_1x_1 + b_1y_1, \qquad \begin{pmatrix} a_1 = \cos\alpha_1 \\ b_1 = \sin\alpha_1 \end{pmatrix},$$

$$y_1 = b_1x + a_1y \qquad y = -b_1x_1 + a_1y_1$$

Similar relations connect L and L_2. Let $(u_1,0)$, (u,v), $(u_2,0)$ be the components with respect to L of the velocities in the three regions marked by their temperatures $T_1 < T < T_2$. The components of the first two velocities with respect to L_1 are

$$(a_1u_1,\ b_1u_1) \quad \text{and} \quad (a_1u - b_1v,\ b_1u + a_1v).$$

The tangential component has the same value on both sides of 1, hence

$$u_1 = \frac{1}{a_1}(a_1u - b_1v), \qquad b_1u_1 = \frac{b_1}{a_1}(a_1u - b_1v).$$

Thus the quotient of the normal velocity components in (T_1) and (T) has the value

$$k_1 = \frac{b_1}{a_1} \cdot \frac{a_1u - b_1v}{b_1u + a_1v}.$$

Again we are going to express everything, including the angles α_1, α_2 of the incident and the reflected waves,

in terms of the temperatures, thus reducing the problem to simple universal algebraic equations once the energy function $U = R.f(T)$ is known. (One might be inclined to consider T_1, T, α_1 as the "natural" data, but then the problem is not algebraic.) The v in a) and b) is now the normal velocity component while v^2 in c) is the square of the total velocity. But because of the continuity of the tangential component, even in c) the v may be interpreted as the normal component. Hence the same relations (3) and (4) as before result for both fronts 1 and 2:

$$D' - D_1 = g(T) - g(T_1), \qquad D'D_1 = TT_1;$$

$$D'' - D_2 = g(T) - g(T_2), \qquad D''D_2 = TT_2.$$

$$(k_1 =) \frac{\rho}{\rho_1} = \frac{D'}{T}, \qquad (k_1^{-1} =) \frac{\rho_1}{\rho} = \frac{D_1}{T_1},$$

$$\frac{p}{p_1} = \frac{D'}{T_1}, \qquad \frac{p_1}{p} = \frac{D_1}{T};$$

$$(k_2 =) \frac{\rho}{\rho_2} = \frac{D''}{T}, \qquad (k_2^{-1} =) \frac{\rho_2}{\rho} = \frac{D_2}{T_2},$$

$$\frac{p}{p_2} = \frac{D''}{T_2}, \qquad \frac{p_2}{p} = \frac{D_2}{T}.$$

For the velocity (u,v) we have the two equations

(6) $$\frac{a_1u - b_1v}{b_1u + a_1v} = \frac{a_1}{b_1} \cdot k_1$$

(7) $$(b_1u + a_1v)^2 - [\frac{b_1}{a_1}(a_1u - b_1v)]^2 = 2(\phi(T_1) \cdot \phi(T)).$$

The first equation is equivalent to the relations

$$a_1u - b_1v = \lambda_1 \cdot a_1k_1 \qquad b_1u + a_1v = \lambda_1 \cdot b_1$$

involving an unknown factor λ_1. They can be solved with respect to u and v as follows:

$$(8) \qquad u = a_1 \cdot \lambda_1 a_1 k_1 + b_1 \cdot \lambda_1 b_1 = \lambda_1 (b_1^2 + a_1^2 k_1) \ ,$$

$$(8') \qquad v = -b_1 \cdot \lambda_1 a_1 k_1 + a_1 \cdot \lambda_1 b_1 = \lambda_1 a_1 b_1 (1 - k_1) \ .$$

The left member of (7) is

$$\frac{v}{a_1^2}[a_1(b_1 u + a_1 v) + b_1(a_1 u - b_1 v)] =$$

$$\frac{v}{a_1^2}(a_1 \cdot \lambda_1 b_1 + b_1 \cdot \lambda_1 a_1 k_1) = \frac{v}{a_1^2} \cdot \lambda_1 a_1 b_1 (1 + k_1) \ ,$$

and if one determines the factor λ_1 by (8'), the equation (7) turns into

$$(9) \qquad \frac{v^2}{a_1^2} \cdot \frac{1 + k_1}{1 - k_1} = 2[\phi(T_1) - \phi(T)] \ ,$$

while (8) and (8') give

$$(10) \qquad \frac{v}{u} = \frac{a_1 b_1 (1 - k_1)}{b_1^2 + a_1^2 k_1} \ .$$

Equations similar to (9) and (10) prevail for the second front 2. Comparison leads to the following two relations for the two unknown angles:

$$\frac{a_1 b_1 (1 - k_1)}{b_1^2 + a_1^2 k_1} = \frac{a_2 b_2 (1 - k_2)}{b_2^2 + a_2^2 k_2} \quad (= \frac{v}{u}) \ ,$$

$$B_1 a_1^2 = B_2 a_2^2 \quad (= v^2).$$

Here we have set

$$A_1 = 2(\phi(T_1) - \phi(T)) = (g(T_1) - g(T)) \cdot (T_1 - T),$$

$$B_1 = A_1 \frac{1 - k_1}{1 + k_1} \ ,$$

while A_2, B_2 have the corresponding meaning. Use the tangents $\frac{b_1}{a_1}$, $\frac{b_2}{a_2}$ as the unknowns t_1, t_2 :

$$\frac{t_1(1 - k_1)}{k_1 + t_1^2} = \frac{t_2(1 - k_2)}{k_2 + t_2^2} ,$$

$$\frac{B_1}{1 + t_1^2} = \frac{B_2}{1 + t_2^2} \quad (= s) .$$

The last equation gives

$$t_1^2 = \frac{B_1}{s} - 1, \qquad t_2^2 = \frac{B_2}{s} - 1 ,$$

and then, after squaring both sides of the first equation one gets

$$\frac{B_1 - s}{(C_1 - s)^2} = \frac{B_2 - s}{(C_2 - s)^2}$$

where

$$C_1 = \frac{B_1}{1 - k_1} = \frac{A_1}{1 + k_1} = \frac{1}{2}(A_1 + B_1) , \quad C_2 = \frac{1}{2}(A_2 + B_2).$$

The resulting equation for s ,

$$(11) \qquad \left(\frac{s - C_2}{s - C_1}\right)^2 = \frac{s - B_2}{s - B_1} ,$$

is not cubic as one might fear, but only quadratic. Setting

$$\frac{s - B_2}{s - B_1} = z$$

we find

$$s = \frac{B_1 z - B_2}{z - 1} , \quad \frac{s - C_2}{s - C_1} = \frac{z - (q_1 z + q_2)}{1 - (q_1 z + q_2)}$$

where

$$\frac{C_1 - B_1}{C_1 - C_2} = q_1 , \quad \frac{C_2 - B_2}{C_2 - C_1} = q_2 .$$

The equation (11) becomes

$$\left(\frac{z - (q_1 z + q_2)}{1 - (q_1 z + q_2)}\right)^2 - 1 = z - 1 \quad ,$$

and after the removal of the foreign factor $z - 1$:

$$(q_1 z + q_2)^2 = z$$

or, finally

(12) $$q_1^2 z^2 - (1 - 2q_1 q_2)z + q_2^2 = 0.$$

Exhibit the inherent symmetry by setting $z_1 = z$, $z_2 = 1/z$:

(12') $$q_1^2 z_1 - (1 - 2q_1 q_2) + q_2^2 z_2 = 0; \quad z_1 z_2 = 1 \quad .$$

Before going on, express the relevant quantities of index 1 in terms of the temperatures T_1, T and the displaced temperatures D_1, D ; one readily finds

$$A_1 = (D_1 - D') + (T_1 - T) \quad ,$$

$$B_1 = (D_1 + D') - (T_1 + T) \quad ,$$

$$C_1 = D_1 - T \quad , \quad q_1 = \frac{D' - T_1}{D_2 - D_1} \quad .$$

Equation (12) is simplified even more by introducing $\sqrt{z} = \zeta$:

$$-\zeta = -\zeta_1 = \frac{s - C_2}{s - C_1} \quad , \qquad -\zeta_2 = \frac{s - C_1}{s - C_2} \quad .$$

It then reads

(13) $$q_1 \zeta_1 + q_2 \zeta_2 = 1, \quad \zeta_1 \zeta_2 = 1 \quad .$$

One easily sees that the sign is correctly determined, namely so that $q_1 \zeta_1 + q_2 \zeta_2$ has the value +1 and not -1.

After z or ζ has been determined one finds

$$v^2 = \frac{B_1 z - B_2}{z - 1} = \frac{B_1 \zeta_1 - B_2 \zeta_2}{\zeta_1 - \zeta_2} \quad ,$$

$$t_1 = \tan \alpha_1 = \left(\frac{B_2 - B_1}{B_1 z_1 - B_2}\right)^{1/2}, \quad -t_2 = \tan \alpha_2 = \left(\frac{B_1 - B_2}{B_2 z_2 - B_1}\right)^{1/2}.$$

For the latter relations we might write

$$\boxed{\begin{aligned} -t_2/t_1 = \zeta_1, \qquad -t_1/t_2 = \zeta_2 , \\ t_1 t_2 = \frac{B_1 - B_2}{B_1 \zeta_1 - B_2 \zeta_2} . \end{aligned}}$$

In order to express v/u we observe that according to our above calculations

$$\frac{t_1^2(1 - k_1)}{t_1^2 + k_1} = \frac{B_1 - s}{C_1 - s} \qquad \text{and}$$

$$\frac{B_2 - B_1}{B_1 - s} = 1 - z \; (= 1 - \zeta^2), \quad \frac{C_2 - C_1}{C_1 - s} = 1 + \quad ,$$

therefore

$$\frac{t_1^2(1 - k_1)}{t_1^2 + k_1} = \frac{B_2 - B_1}{C_2 - C_1} \cdot \frac{1}{1 - } \quad ,$$

and as

$$\frac{B_2 - B_1}{C_2 - C_1} = 1 - q_1 - q_2 \tag{14}$$

and $(1 - \zeta)t_1 = t_1 + t_2$, the expression $\dfrac{t_1(1 - k_1)}{t_1^2 + k_1}$

for v/u may be written in the more symmetric form

$$\boxed{\frac{v}{u} = \frac{1 - q_1 - q_2}{t_1 + t_2} .}$$

§ 2. Discussion of signs. The qualitative discussion of the results is more difficult than the straightforward algebraic part. B_1, B_2, q_1, q_2 are positive, $C_2 > 0$, $C_1 < 0$. A first condition for the existence of a solution corresponding to the temperatures

T_1, T, T_2 is the reality of the roots of the equation (12) or (13), namely

(15) $$Q = 1 - 4q_1q_2 > 0.$$

But this is not quite enough; for at least one of the roots $z = z'$, z'' of (12) the quotient $\frac{B_2 - B_1}{B_1z - B_2}$ must be positive. Exactly <u>one</u> of the quantities $B_1z' - B_2$, $B_1z'' - B_2$ will have the sign of $B_2 - B_1$, if their signs are opposite, or if

(16) $$q_1^2(B_1z' - B_2)(B_1z'' - B_2) < 0.$$

The left side equals

$$q_1^2[B_1^2z'z'' - B_1B_2(z' + z'') + B_2^2] = B_1q_2^2 - B_1B_2(1 - 2q_1q_2) + B_2q_1^2,$$

and hence the relation (16) reduces to

(17) $$\boxed{J \equiv (B_1q_2 + B_2q_1)^2 - B_1B_2 < 0 \quad .}$$

If z', z'' are conjugate complex, (16) is necessarily positive. Consequently (17) implies (15); the relation

(18) $$(B_1q_2 + B_2q_1)^2 - (B_1q_2 - B_2q_1)^2 = 4q_1q_2B_1B_2$$

confirms this in a more explicit way. Thus the case of <u>one solution</u> is characterized by $J < 0$.

If, however

(19) $$J > 0, \quad Q > 0$$

we may have no or two solutions. The left side of (16) is now positive, therefore

either $B_1z' < B_2$, $B_1z'' < B_2$ and thus $B_1^2z'z'' < B_2^2$

or $B_1z' > B_2$, $B_1z'' > B_2$ and thus $B_1^2z'z'' > B_2^2$.

The alternative takes on the form

$$B_1^2 q_2^2 - B_2^2 q_1^2 \lesseqgtr 0 \quad \text{or} \quad B_1 q_2 - B_2 q_1 \lesseqgtr 0.$$

Result: In case (19) we have <u>two solutions or no solutions</u> according to whether

$$B_1 q_2 - B_2 q_1 \quad \text{and} \quad B_1 - B_2$$

have equal or opposite signs. We add the observation that, due to the relation (17) or

$$(B_1 q_2 - B_2 q_1)^2 = J + Q B_1 B_2 \quad ,$$

$B_1 q_2 - B_2 q_1$ can not change sign in an interval where (18) holds. Moreover (14) shows that $B_2 > B_1$ implies $q_1 + q_2 < 1$ and a fortiori $Q > 0$.

The difference $\frac{B_1}{q_1} - \frac{B_2}{q_2}$, the sign of which plays a role in this discussion, is, by definition

$$(D_2 - D_1)\left[2 - \frac{T - D_1}{D' - T_1} - \frac{D_2 - T}{T_2 - D''}\right] \quad .$$

But

$$\frac{T}{D'} = \frac{D_1}{T_1} \quad , \qquad \frac{D_2}{T_2} = \frac{T}{D''} \quad ,$$

hence we can write more simply

$$\frac{1}{D_2 - D_1}\left(\frac{B_1}{q_1} - \frac{B_2}{q_2}\right) = 2 - \left(\frac{D_1}{T_1} + \frac{D_2}{T_2}\right) = 2 - \left(\frac{T}{D'} + \frac{T}{D''}\right) \quad ,$$

and the alternative $\frac{B_1}{q_1} - \frac{B_2}{q_2} \lesseqgtr 0$ amounts to

$$\boxed{\frac{D_1}{T_1} + \frac{D_2}{T_2} \gtreqless 2 \ .}$$

We proceed to evaluate this general analysis.

Head-on collision is characterized by $B_1 = B_2$, or $q_1 + q_2 = 1$, which gives $J = 0$. Grazing wave ($t_1 = \infty$) is indicated by $B_1 z - B_2 = 0$, or $z = B_2/B_1$ must satisfy the equation (12), which again gives $J = 0$. Hence J should

split into two factors, one being $B_1 - B_2$. Indeed, using the expression

$$q_1 = \frac{C_1 - B_1}{C_1 - C_2} , \qquad q_2 = \frac{C_2 - B_2}{C_2 - C_1}$$

one easily gets

$$J = \frac{(B_1 - B_2)(B_1 C_2^2 - B_2 C_1^2)}{(C_2 - C_1)^2} .$$

Hence the sign of J depends on those of the differences

$$B_2 - B_1 \qquad \text{and} \qquad \frac{C_2^2}{B_2} - \frac{C_1^2}{B_1} .$$

For a given T we plot $B_2 = B(T_2)$ and $\frac{C_2^2}{B_2} = \Gamma(T_2)$ as functions of T_2. For small $T_2 - T = \Delta T$ one finds

$$B(T_2) \sim f'(f' + 1) \cdot \frac{(\Delta T)^2}{T} , \quad \Gamma(T_2) \sim \frac{f'+1}{f'} T + Wf(T) \frac{\Delta T}{f'^2}$$

Notice the differential expression Wf in the formula for $\Gamma(T_2) - \Gamma(T)$. Th kes it plausible that $\Gamma(T_2)$ is a monotone increasing function of T_2, or at least that

(H_2): $\boxed{\Gamma(T_2) - \Gamma(T) \quad \underline{\text{has the sign of}} \quad T_2 - T.}$

The behavior at the ends $T_2 = 0$ and $T_2 \to \infty$ supports our belief in the monotony of $\Gamma(T_2)$; indeed for small T_2

$$\Gamma(T_2) \sim \frac{T^2}{2f(T)}\left(1 + \frac{f'(0)}{f(T)} \cdot T_2\right) ,$$

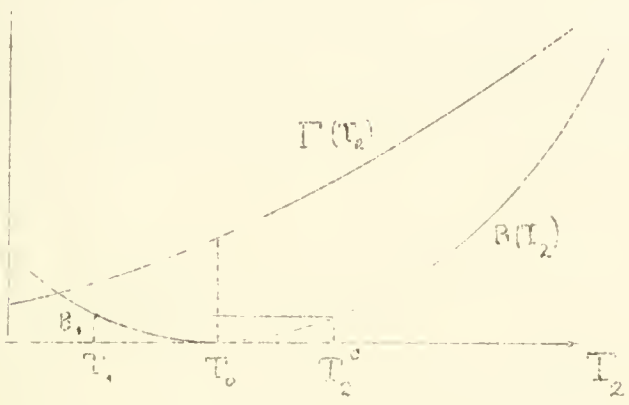
and $\Gamma \to \infty$ with $T_2 \to \infty$. (H_2) admitted, $\frac{C_2^2}{B_2} - \frac{C_1^2}{B_1}$ will always be positive for $T_1 < T < T_2$, and <u>a grazing or nearly grazing wave proves impossible.</u> The alternative $J \lesssim 0$ reduces to the simpler $B_2 \lesssim B_1$.

$B(T_2)$ behaves in the neighborhood of $T_2 = T$

according to the formula

$$B(T_2) \sim f'(f' + 1) \cdot \frac{(\Delta T)^2}{T} .$$

One will therefore expect that the graph of $B(T_2)$ has the shape of a trough. $B(0) = 2f(T)$ and $B(T_2)$ tends to infinity with $T_2 \to \infty$, under the hypothesis that $f(T) \to \infty$ for $T \to \infty$.



Given T_1, T, the horizontal through the point $(T_1, B(T_1))$ on the graph of $B(T_2)$ determines a value $T_2^o > T$ for which $B(T_2) = B(T_1) = B_1$. For all $T_2 > T_2^o$ the condition $B_2 > B_1$ is fulfilled which indicates that <u>the problem has one solution.</u>

In the interval $T < T_2 < T_2^o$ where $B_2 < B_1$, $J > 0$, one has to discuss the signs of

$$Q = 1 - 4q_1q_2 \quad \text{and} \quad L = 2 - \left(\frac{D_1}{T_1} + \frac{D_2}{T_2}\right) ,$$

$Q > 0$ and $L < 0$ being the criteria for <u>two</u> solutions. Let us therefore investigate $1 - 4q_1q_2$ or

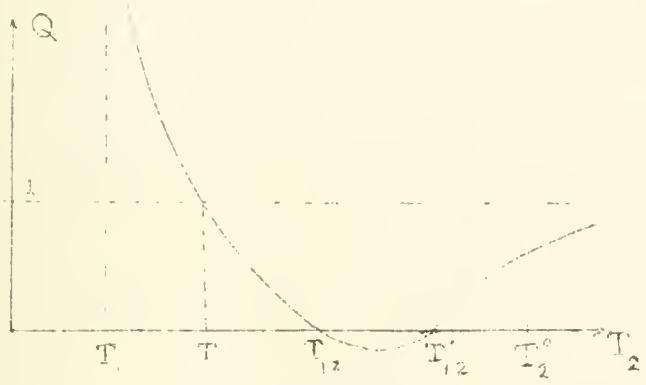
$$(2\text{@}) \qquad (D_2 - D_1)^2 - 4(D' - T_1)(T_2 - D'')$$

for T_1, T_2 near T. Setting $T_1 = T(1 - x)$, $T_2 = T(1 + y)$ one obtains in first approximation

$$T^2(f' + 1)^2(x - y)^2 \ (\geqq 0).$$

Were the quadratic form on the left positive-definite, one would expect Q to be pos itive throughout. But it is

only semi-definite, with the neighborhood of the diagonal $x = y$ as the danger zone. If one sets $y - x = \frac{2t}{f' + 1} \cdot x^2$ with an arbitrary coefficient t, then (20) becomes T^2x^4 times a quadratic form of t, the discriminant of which once more involves our familiar Wf, namely equals $2(1 + f')Wf$. Hence this form is indefinite and Q becomes actually negative in the neighborhood of the diagonal $x = y$. The values of Q for $T_2 = T_1, T, \infty$ are $\infty, 1, 1$, respectively, and for given T_1, T we expect a graph of $Q(T_2)$ as shown in the figure. The interval (T_{12}, T'_{12}) in which $Q < 0$ is part of (T, T_2^0).



$L(T_2)$ has a constant sign throughout the piece (T, T_{12}), and since $D_1/T_1 < 1$ and $D_2/T_2 = 1$ for $T_2 = T$, it is positive there and hence <u>no</u> solution exists. For $T'_{12} < T_2 < T_2^0$ one will have two solutions, if the constant sign of L in this interval is $-$. Thus the question arises whether L changes sign before T_2 reaches T_2^0. For a given $T_1 = T(1 - x)$ near T the expression $L(T_2)$ actually changes sign at a $T_2 = T(1 + y)$ for which $y - x$ is of the order x^2, and B_2 will be less than B_1 at this T_2, provided

(H^*): $\qquad W^*f = \frac{1}{2}[f'(f' + 1)(2f' - 1) + f''T] > 0.$

If we make this further hypothesis, L turns out to be <u>negative</u> in the interval $T'_{12} < T_2 < T_2^0$ for sufficiently

small $T - T_1$ and hence, for reasons of continuity, will keep this sign under all circumstances: there are then two solutions in that interval.

The fact that $L < 0$ or $B_1 q_2 < 0$ for $T_2 = T_2^o$ $(B_2 = B_1)$ implies that there $q_1 - q_2 > 0$ or

$$(20a) \qquad (D_2 - D_1)(q_1 - q_2) = (D' + D'') - (T_1 + T_2) > 0.$$

For the same value of T_2 we have $q_1 + q_2 = 1$, and hence

$$\zeta_1 = 1, \quad \zeta_2 = 1; \qquad \zeta_1 = q_2/q_1, \quad \zeta_2 = q_1/q_2$$

are the two solutions of (13). In view of the general formulas

$$(20b) \qquad (\tan \alpha_1)^2 = \zeta_2 \frac{B_2 - B_1}{B_1 \zeta_1 - B_2 \zeta_2}, \quad (\tan \alpha_2)^2 = \zeta_1 \frac{B_2 - B_1}{B_1 \zeta_1 - B_2 \zeta_2}$$

the second solution gives in the neighborhood of $T_2 = T_2^o$:

$$\alpha_1 \sim \frac{q_1}{\sqrt{B_1}} \cdot \sqrt{\frac{B_1 - B_2}{q_1^2 - q_2^2}} \qquad \alpha_2 \quad \frac{q_2}{\sqrt{B_2}} \cdot \sqrt{\frac{B_1 - B_2}{q_1^2 - q_2^2}} \quad .$$

Being real for $T_2 < T_2^o$ it ceases to exist beyond T_2^o. The first solution must therefore be the one which continues beyond T_2^o. For an infinitesimal $B_2 - B_1$ set $\zeta_1 \sim 1 + \epsilon$ (ϵ infinitesimal), consequently

$$\zeta_2 \sim 1 - \epsilon \quad \text{and} \quad q_1(1 + \epsilon) + q_2(1 - \epsilon) = 1 \quad \text{or}$$

$$(q_1 - q_2)\epsilon = 1 - q_1 - q_2 = \frac{B_2 - B_1}{D_2 - D_1} \quad .$$

Thus one derives from (20b) for the first solution:

$\alpha_1 = \alpha_2 = \alpha^o$ at $T_2 = T_2^o$ where

$$(\tan \alpha^o)^{-2} = \frac{(B_1 + B_2) - (D_2 - D_1)(q_1 - q_2)}{(D_2 - D_1)(q_1 - q_2)} \quad .$$

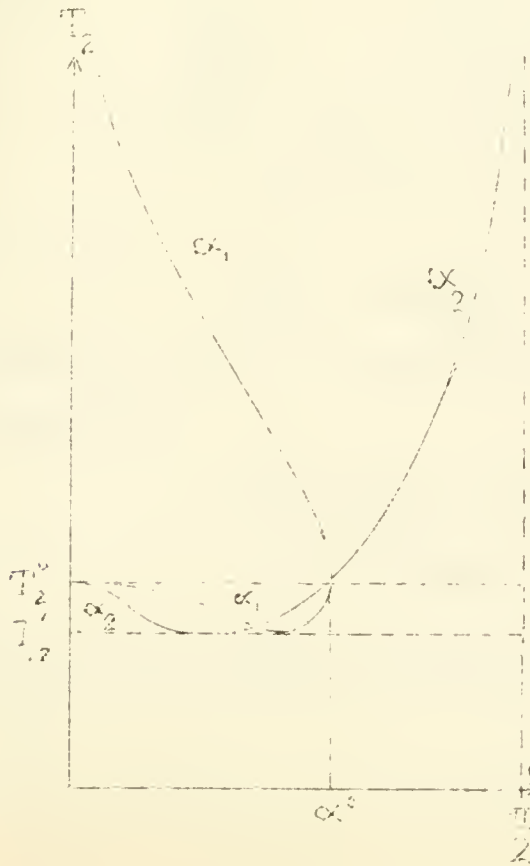
By (20a) and the expression of B_1 and B_2 this formula reduces to

$$\tan\alpha^{\circ} = \sqrt{\frac{D' + D'' - T_1 - T_2}{D_1 + D_2 - 2T}} \quad (T_2 = T_2^{\circ})$$

If T_2 tends to infinity then $q_1 \to 0$ and $\zeta_1 \to \infty$ in such a manner that

$$q_1 \zeta_1 \to 1 : \quad \zeta_1 \sim 1/q_1 , \quad \zeta_2 \sim q_1$$

Hence t_1^2 and t_2^2 tend to zero and infinity at least as strongly as T_2^{-1} and $f(T_2)$ respectively. The limit is the strange phenomenon of a head-on impact ($\alpha_1 \to 0$) that slides off ($\alpha_2 \to \frac{\pi}{2}$) instead of being thrown back and thereby acquires infinite strength ($p_2/p = D_2/T \to \infty$). The fact that $t_1 \to 0$ with $T_2 \to \infty$ shows that α_1 never exceeds a certain maximum value $\overline{\alpha_1} < \frac{\pi}{2}$. The curves show the qualitative behavior and are in accordance with v. Neumann's results for constant specific heat. In Nature the Mach phenomenon of detachment bridges the gap $\overline{\alpha_1} < \alpha_1 < \frac{\pi}{2}$ and cuts off the top part of the picture corresponding to temperatures T_2 above a certain limit.



Our analysis reveals the importance of the condition $Wf > 0$. The hypothesis (H^*)

plays a more accessory role; as a matter of fact, the situation would be simpler if $W^{*}f < 0$, because then only the branch $T_2 > T_2^0$ of solutions would exist.

C. Proof of the hypotheses (H_1) and (H_2)

1. Qualitative character of the adiabatic curves.

We propose to show that the inequality

$$Wf > 0$$

is equivalent to the fact that the adiabatic curves in the V,p - diagram are convex:

$$\left(\frac{d^2p}{dV^2}\right)_{adab.} > 0.$$

Indeed this proposition is an immediate consequence of the following simple formula:

$$\left(\frac{d^2p}{dV^2}\right)_{ad} = 2 \cdot \frac{Wf(T)}{(f'(T))^3} \cdot \frac{p}{V^2} \quad . \tag{21}$$

Proof. Along an adiabatic curve

$$Rf'(T).dT + p.dV = 0 \qquad \text{or}$$

$$\frac{dT}{dV} = - \frac{p}{R \cdot f'(T)} \quad . \tag{22}$$

Because of $RT = pV$ the same differential equation may be written as

$$f'(T)(p.dV + V.dp) + p.dV = 0 \qquad \text{or}$$

$$- \frac{dp}{dV} = \frac{f' + 1}{f'} \cdot \frac{p}{V} = R \cdot \frac{f' + 1}{f'} T \cdot \frac{1}{V^2} \quad . \tag{23}$$

Hence by logarithmic derivation with respect to V,

$$\frac{d^2p}{dV^2} \Big/ \frac{dp}{dV} = \left(\frac{f''}{f'+1} - \frac{f''}{f'} + \frac{1}{T}\right)\frac{dT}{dV} - \frac{2}{V}$$
$$= \frac{-f''\cdot T + f'(f'+1)}{f'(f'+1)\cdot T}\cdot\frac{dT}{dV} - \frac{2}{V},$$

or, after inserting (22) and substituting p V for RT in the denominator,

$$\left(\frac{f''\cdot T - f'(f'+1)}{f'^2(f'+1)} - 2\right)\frac{1}{V} = -\frac{2}{V}\cdot\frac{Wf}{f'^2(f'+1)}$$

Multiplication by (23) yields the desired result (21).

2. Proof of the hypothesis (H_1). With a given function f(T) one can form the skew-symmetric function $f(T_2) - f(T_1)$. It has the sign of $\Delta T = T_2 - T_1$ for an infinitesimal ΔT if and only if $f' > 0$. A classical theorem of Calculus states that this condition $f'(T) > 0$ implies

$$f(T_2) - f(T_1) > 0 \quad \text{for} \quad T_2 > T_1$$

even for arbitrary finite differences ΔT.

Let f(T) be any monotone-increasing function of the positive argument T. We are going to prove an analogous theorem concerned with a much more complicated skew-symmetric function $\eta(T_1, T_2)$ formed by means of f(T), namely

$$\eta(T_1, T_2) = \int_{T_1}^{T_2}\frac{f'(T)}{T}dT - \log\frac{D_2}{D_1} = \int_{T_1}^{T_2}\frac{f'(T)}{T}dT - \log\frac{D_2}{D_1}$$

where D_1, D_2 are defined by the relations (3), g(T) = T + 2f(T). It has been seen that this η for infinitesimal

$\Delta T = T_2 - T_1$ is of the order $(\Delta s)^3$, and the heat production sign of ΔT if and only if

$$2 \cdot Wf = f'(f' + 1)(2f' + 1) - Tf'' > 0.$$

We maintain that $Wf(T) > 0$ for all T implies $\eta(T_1, T_2) > 0$ for all T_1, T_2, $T_2 > T_1$. The question asked at the end of A., § 1, is answered thereby in the affirmative.

The proof is suggested by the study of a linear shock wave with the temperatures T_1, T_2. In purely mathematical disguise it runs as follows.

Since $Wf(T_1) > 0$ implies $\eta(T_1, T_2) > 0$ for $T_2 > T_1$ sufficiently near to T_1, the function $\eta(T_1, T_2)$ of T_2 could never become negative unless it passes through zero for some definite value $T_2 > T_1$ of the argument. We will show that this relation

$$\eta(T_1, T_2) = 0 \tag{24}$$

leads to a contradiction.

Choose two positive numbers p_1, V_1, determine R by $p_1 V_1 = RT_1$ and, in agreement with the scheme (4), p_2, V_2 by

$$\frac{p_2}{p_1} = \frac{D_2}{T_1} = \frac{T_2}{D_1}, \qquad \frac{V_2}{V_1} = \frac{D_1}{T_1} = \frac{T_2}{D_2}$$

These relations imply

$$\frac{p_2V_2}{p_1V_1} = \frac{T_2}{T_1} \quad , \qquad \text{hence} \quad p_2V_2 = RT_2 .$$

The hypothetic equation (24) may then be written

$$\varphi(T_1, T_2) = \int_{T_1}^{T_2} \frac{f'(T)}{T} dT + \log \frac{V_2}{V_1} = 0 .$$

Form the following "entropy" function η^* of p, V:

$$\eta^* = \int_{T_1}^{T_2} \frac{f'(x)}{x} dx + \log \frac{V}{V_1} \quad \text{where } T = \tfrac{1}{2}pV .$$

It has been shown in §1, that along the "adiabatic lines" $\eta^*(p, V) = \text{const.}$ one has

$$\frac{dp}{dV} < 0, \qquad \frac{d^2p}{dV^2} > 0 .$$

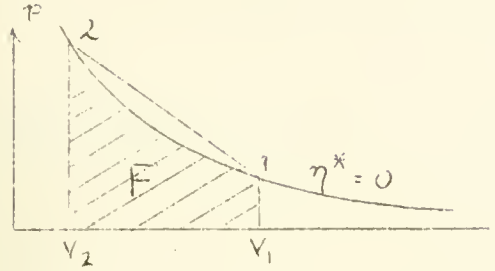
(24) can now be interpreted as the statement that the two points (V_1, p_1) and (V_2, p_2) lie on the same adiabatic $\eta^* = 0$. Because of $T_2 > T_1$ we have $V_2 < V_1$. Along the adiabatic $\eta^* = 0$ the following differential equation obtains

$$f'(T)\cdot dT + \frac{T}{V} dV = 0 \quad \text{or} \quad Rf'(T)\, dT + p\cdot dV = 0$$

and thus we find by integration

$$R\left(f(T_2) - f(T_1)\right) = \int_2^1 p\, dV$$

The right side is the shaded area F below the adiabatic



$\eta^* = 0$. However, there is a second more direct way of determining the difference $R(f(T_2) - f(T_1))$. Indeed, $2(f(T_2) - f(T_1)) = g(T_2) - g(T_1) - T_2 + T_1 = D_2 - D_1 - \frac{D_1 D_2}{T_1} + T_1 = T_1\left(1 + \frac{D_2}{T_1}\right)\left(1 - \frac{D_1}{T_1}\right)$.

$$R(f(T_2) - f(T_1)) = \frac{1}{2} p_1 V_1 \left(1 + \frac{p_2}{p_1}\right)\left(1 - \frac{V_2}{V_1}\right) \quad \text{or}$$

$$R(f(T_2) - f(T_1)) = \frac{1}{2}(p_1 + p_2)(V_1 - V_2)$$

(Hugoniot equation). Thus we have found two values for $R(f(T_2) - f(T_1))$: the area of F and the areaof the trapezoid obtained from F by straightening the upper boundary 21. The contradiction is obvious; because of the convexity of the curve the second area is larger than the first.

§ 3. Proof of hypothesis (H_2). If we write T_1, T_2 instead of T, T_2 the proposition (H_2) concerns the difference

$$\frac{(D_2 - T_1)^2}{D_1 + D_2 - T_1 - T_2} - \frac{f'(T_1) + 1}{f'(T_1)} \cdot T_1 .$$

But

$$T_1(D_1 + D_2 - T_1 - T_2) = T_1 D_1 + T_1 D_2 - T_1^2 - D_1 D_2$$
$$(D_2 - T_1)(T_1 - D_1).$$

Thus, our task is to prove:

$$\chi(T_1, T_2) = \frac{D_2 - T_1}{T_1 - D_1} - \frac{f'(T_1) + 1}{f'(T_1)}$$

has the sign of $T_2 - T_1$, provided $Wf > 0$.

This time we are not dealing with a skew-symmetric function of T_1, T_2, and we shall see that the situation for $T_2 > T_1$ is different from $T_1 < T_2$. Nevertheless we can give a proof holding uniformly for both cases.

Once for all we suppose $Wf > 0$. Considering that our calculations in B., § 2, have established our theorem for T_2 sufficiently near to T_1, it is sufficient to show that the equation $\chi(T_1, T_2) = 0$ can never hold. The proof is indirect: we assume that we are in possession of two different values T_1, T_2 for which

$$\chi(T_1, T_2) = 0 \quad .$$

We choose and determine the numbers p_1, V_1, p_2, V_2, R as before. Then (25) gives

$$\frac{p_2 - p_1}{V_1 - V_2} = \frac{f'(T_1) + 1}{f'(T_1)} \cdot \frac{p_1}{V_1}$$

We now introduce the straight line through the point (V_1, p_1) defined by

$$\frac{p - p_1}{V_1 - V} = \frac{f'(T_1) + 1}{f'(T_1)} \cdot \frac{p_1}{V_1} \quad .$$

Because of the equation (23) this is the tangent of the adiabatic $\eta^* = 0$ at the point (V_1, p_1). Our hypothetical equation (25) states that the point (V_2, p_2) lies on this tangent.

Integrating the relation

$$RT \cdot d\eta^* = Rf'(T) \cdot dT + p \cdot dV$$

along the segment 12 of our tangent, we get

$$R \int_1^2 f'(T) \cdot dT = R(f(T_2) - f(T_1))$$

and

$$\int_1^2 p \cdot dV = - \int_2^1 p \, dV = - \frac{1}{2}(p_1 + p_2)(V_1 - V_2).$$

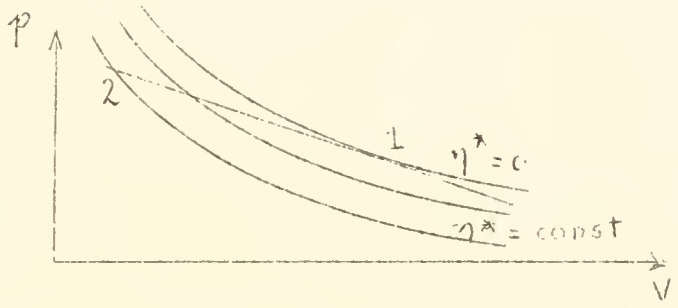
According to the Hugoniot equation these two expressions cancel one another; therefore

$$\int_1^2 T \cdot d\eta^* = 0. \tag{26}$$

But because of the convexity of all adiabatic lines, η^* varies monotonely (namely decreases) when one moves along the tangent starting from 1, whether in the one direction or the other. This circumstance renders the equation (26) impossible. (See figure on next page).

Moving along the tangent in the direction of increasesing T (or decreasing V) one comes to lower values

of entropy, whereas we already know that in the case $T_2 > T_1$ the point 2 should correspond to a higher entropy than 1. But for the case $T_2 < T_1$ the more refined argument using the equation (26) seems necessary.



In view of the fact that $T_1 - D_1$ is positive or negative according to whether $T_2 \gtrless T_1$, we can give our inequality the simple form

$$(D_1 + D_2 - 2T_1)\cdot f'(T_1) > T_1 - D_1 .$$

Interchanging the notations T_1 and T_2 one finds also

$$(2T_2 - D_1 - D_2)\cdot f'(T_2) < D_2 - T_2 .$$

§4. Concluding remarks. The third consequence which one should be able to draw from the condition $Wf > 0$ is the fact that, given T_1 and T there is a single interval of the T_2 - axis (contained in the interval $T < T_2 < T_2^o$) where $Q = 1 - 4q_1q_2$ becomes negative. But I have not yet completely succeeded in settling this point.

Instead, I propose to show by a quite trivial argument that $B(T_2)$ is of trough-like shape, more exactly:

Given T_1, the positive function

$$B = D_1 + D_2 - T_1 - T_2$$

of T_2 is increasing for $T_2 > T_1$, decreasing for $T_2 < T_1$,

Let us complute the derivative ' with respect to T_2,

$$B' = D_1' + D_2' - 1 .$$

The equations

$$D_1 D_2 = T_1 T_2 , \qquad D_2 - D_1 = g(T_2) - g(T_1)$$

yield

$$D_1' D_2 + D_2' D_1 = T_1 , \quad D_2' - D_1' = g'(T_2),$$

consequently

$$D_1'(D_1 + D_2) = T_1 - D_1 g'(T_2).$$

$$D_2'(D_1 + D_2) = T_1 + D_2 g'(T_2),$$

$$(D_1' + D_2' - 1)(D_1 + D_2) = (2T_1 - D_1 - D_2) +$$

$$+ (D_2 - D_1)(1 + 2f'(T_2)) = 2[(T_1 - D_1) + (D_2 - D_1)f'(T_2)]$$

This formula puts the sign of the derivative in evidence.

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